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THE BACKWARD RECURRENCE METHOD FOR COMPUTING THE REGULAR BESSEL FUNCTION

by Thomas E. Michels

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Greenbelt, Maryland

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SUMMARY

The Backward Recurrence Method (suggested by Dr. J. C. P. Miller) for computing the regular Bessel function is investigated for real, imaginary, and complex arguments. From this, one can obtain the functions $J_n(z)$, $I_n(z)$, $\mathrm{ber}_n(z)$, and $\mathrm{bei}_n(z)$. A section has also been devoted to show the behavior of the functions for all arguments, and graphs are presented which show the magnitudes of the functions for all orders and real arguments between 0 and 6000.

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(Manuscript Received August 19, 1963)

by

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INTRODUCTION

Bessel's Equation is a second order differential equation with a singular point at $z = 0$ and is of the form

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{n^2}{z^2}\right)y = 0 .$$

Its two independent solutions are referred to as the regular and irregular Bessel functions (see Appendix A). Computation of the irregular function, $Y_n(z)$, is fairly easy, since one can use a straightforward recurrence relationship once he has two starting values (Reference 1) and obtain good accuracy for all n and z values. However, this method cannot be used to compute the regular function, $J_n(z)$, due to error accumulation in the recurrence. For large order and argument, asymptotic formulas prove to be too cumbersome; and in computer computation they are also inaccurate. But a method was devised by Dr. J. C. P. Miller (Reference 2) whereby an ingenious use of the recurrence relationship yields good values for all orders and arguments which fall within the limits of any particular computer.

It is the purpose of this report to explain the method and the need for it, as well as give details on the behavior and use of the recurrence formula and the accuracies obtained. (Hopefully it will also give insight into the behavior of the functions themselves.) This has been done for the argument z real, imaginary, and complex. In order to keep to this primary purpose, not much will be said about the irregular function. In general, however, the irregular functions behave just oppositely to the regular function in the range where $J_n(z)$ decreases or increases exponentially, while they behave similarly in the rest of the range. That is, if the magnitude of $J_n(z)$ *decreases exponentially*, the irregular function *increases;* and when $J_n(z)$ oscillates, the irregular function does the same.

It should be noted that the recurrence formula computes a sequence of Bessel functions as a function of n, the order, rather than the usual tabulations which are a function of z. Throughout this discussion, where the order is referred to as n in $J_n(z)$, it should be taken to mean any value of n (integer, half orders, one-third orders, etc.), unless otherwise specified. Also m is always greater than n, and $z\ (= x + iy)$ is regarded as a complex number. If just the real numbers are being referred to, x will be used, and for the imaginary iy will be used.

Reference is made from time to time to single and double precision. All tests were made on an IBM 7090 computer which has a word length of 36 binary bits. In the floating point mode, single precision arithmetic is done with 9 decimal digits and double precision with 18 decimal digits. This is implied throughout.

EXPLANATION OF THE METHOD FOR COMPUTING THE REGULAR BESSEL FUNCTION

Importance of the Backward Recurrence

It is well known that both the regular $J_n(z)$ and the irregular $Y_n(z)$, Bessel functions (see Appendix A) obey the recurrence relationship

$$F_{n+1}(z) + F_{n-1}(z) = \frac{2n}{z} F_n(z) .$$

If the recurrence is started with either $J_0(z)$, $J_1(z)$ or $Y_0(z)$, $Y_1(z)$, in the computer where there are a fixed number of digits, good accuracy is obtained for $n \leq z$ for both functions. This accuracy continues for the irregular function since $Y_n(z)$ *increases exponentially* for $n > z$, and the number of accurate digits is maintained or increased; but for the regular function at this point accuracy is lost very rapidly, since $J_n(z)$ *decreases* exponentially for $n > z$.

Actually, throughout the recurrence, by recurring in either direction, we obtain numbers which represent solutions to Bessel's equation, but they are linear combinations of the regular and irregular solution. Therefore, the error introduced in recurring for the regular function is actually some constant times the irregular function.

In using the formula for recurring Bessel functions, and recurring in the direction of increasing n, the number of digits of accuracy is lost very rapidly. This is not even taking into account loss of accuracy due to round off.

Let us take an extreme case and look at it this way: Assume we are in a computer which produces 8 significant digits and we are using the recurrence formula below with a small error in each term on the right, say in the last decimal place. We ask what the error ϵ_{n+1} is in our new term F_{n+1}:

$$F_{n+1}(z) + \epsilon_{n+1} = \frac{2n}{z} \left[F_n(z) + \epsilon_n\right] - \left[F_{n-1}(z) + \epsilon_{n-1}\right] .$$

Let us assume reasonable values—say for the second term on the right $F_{n-1}(z) \approx 10^{-1}$ — and make the error the best we could have, and therefore in the last significant digit, or $\epsilon_{n-1} \approx 10^{-8}$. Let the magnitude of $F_n(z)$ be smaller, say 10^{-2}, again with the error being in the last place, or

$\epsilon_n \approx 10^{-9}$. Now we recur to a smaller number $F_{n+1}(z)$, say of the magnitude 10^{-3}. Still, there was an error at 10^{-8}, and therefore $\epsilon_{n+1} \approx 10^{-8}$. The $(n-1)$ term had 7 accurate digits and two terms away, at $(n+1)$, the number of accurate digits is reduced by two and our error is now in the sixth significant digit.

A similar argument can be made to show that when we recur to numbers of increasing magnitude, the number of significant digits of accuracy is actually increased, or at least stays equal to the maximum number of digits available in the computer. Therefore, the use of the foregoing recurrence formula to compute the regular Bessel functions in the region where they decrease in magnitude is not acceptable as it stands.

Statement of Method

Miller's Method

A scheme was devised by Miller (Reference 2) whereby the recurrence formula could be used and accuracy still maintained throughout the whole range of n values. We start at some m, with

$$F_{m+1}(z) = 0 ,$$

$$F_m(z) = a ,$$

where a is any constant, and use the recurrence formula, but in *decreasing* magnitude in n (where we are recurring to larger numbers and therefore increasing accuracy) and generate a series of functions $F_n(z)$, $F_{n-1}(z)$, $\cdots F_0(z)$ at some $n < m$, by

$$F_{n-1}(z) = \frac{2n}{z} F_n(z) - F_{n+1}(z)$$

which are all a constant multiple of the regular Bessel function. That is,

$$F_n(z) = \alpha J_n(z) .$$

Proof

Since the recurrence formula follows for both the regular and irregular function, we can say that it follows for a linear combination of them also, or

$$F_n(z) = \alpha J_n(z) + \beta Y_n(z) .$$

Since

$$F_{m+1}(z) = 0 = \alpha J_{m+1}(z) + \beta Y_{m+1}(z) ,$$

we have

$$F_n(z) = \alpha\left[J_n(z) - J_{m+1}(z)\frac{Y_n(z)}{Y_{m+1}(z)}\right] .$$

Therefore at some $n < m$

$$J_{m+1}(z)\frac{Y_n(z)}{Y_{m+1}(z)} \approx 0$$

and hence

$$F_n(z) = \alpha J_n(z)$$

to any desired degree of accuracy. Therefore all that remains to obtain the regular Bessel function is to determine α.

Determination of α

There are a number of ways in which α can be determined. One method is to use an addition formula such as the following (Reference 3):

$$2^{\nu}\sum_{k=0}^{\infty}(\nu + 2k)J_{\nu+2k}(z)\,z^{-\nu}\frac{\Gamma(\nu + k)}{k!} = 1 ,$$

which, of course, for integral orders or $\nu = 0$, becomes

$$J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z) = 1 ,$$

or, for $F_k(z) = \alpha J_k(z)$,

$$F_0(z) + 2\sum_{k=1}^{\infty} F_{2k}(z) = \alpha .$$

The summation can be carried out to $k = n$ and will give α the particular accuracy desired. Here n implies the number of the term $< m$ where $F_n(z) = \alpha J_n(z)$ (see page 7).

Of course, if an expression is known for a particular $J_n(z)$ value, say

$$J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\sin z ,$$

then α is more easily determined by

$$\alpha = \frac{F_{1/2}(z)}{J_{1/2}(z)} .$$

For z real, the foregoing addition formula proves to be fairly accurate in the determination of α; however, for z complex or imaginary, using the addition formula in the computer proves to be inaccurate (see page 8).

The value of the Bessel function is only as accurate as α, since each term is divided by α; therefore the computation of α is extremely critical.

Error Introduced from Addition Formulas for Complex or Imaginary Argument

In looking at the regular Bessel function of complex or imaginary argument, we observe that

$$|J_0(z)| \approx 0.1\, e^y$$

and

$$\lim_{n\to\infty} J_n(z) = 0 .$$

As n increases, $J_n(x+iy)$ decreases at an exponential rate almost immediately for $|z| < 10$, and for larger z it takes a few more terms before the exponential decrease. It is almost a logarithmic spiraling into the origin of the complex plane.

With this in mind, assume we are performing the addition formula for integral orders. We want to investigate the error that will exist in the determination of α, assuming an error in some $F_n(z)$.

In our downward recurrence we have the following:

$$F_0(z) + \epsilon_0 + 2\sum_{n=1}^{\infty}\left(F_{2n}(z) + \epsilon_{2n}\right) = (1+\epsilon)\alpha .$$

Assume the error to be in the last digit in all $F_n(z)$, which of course is the best we could expect in the computer. Upon factoring out α, we have

$$\alpha\left(J_0(z) + \epsilon_0\right) + 2\alpha\sum_{n=1}^{\infty}\left(J_{2n}(z) + \epsilon_{2n}\right) = \alpha(1+\epsilon) ;$$

and since

$$J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z) = 1 ,$$

we have

$$\epsilon_0 + 2 \sum_{n=1}^{\infty} \epsilon_{2n} = \epsilon ,$$

where ϵ is now the error we introduce in computing the number 1.

Since the errors decrease at an exponential rate as n increases,

$$\epsilon \approx \epsilon_0 ,$$

which says the error we introduce in computing the digit 1 is of the order of the error in $J_0(z)$. Therefore α will be in error by this factor. It can be seen that if $J_0(z)$ had five figure accuracy and was of the order of magnitude 10^7, the summation would actually become meaningless.

Thus, to determine α one of the Bessel function values must be computed. This can be done by an asymptotic formula which simplifies for $J_0(z)$, or the following integral relationship can be used:

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z \sin\theta)\, d\theta ,$$

or

$$J_0(z) = \frac{1}{\pi} \int_0^{\pi} \cos(z \sin\theta)\, d\theta .$$

Integration of the expression for $J_0(z)$ by the trapezoidal rule generally yields sufficient accuracy, especially for large z values for which the slopes of the functions become large. For larger z values, of course, $\Delta\theta$ must be decreased. Integration performed by using the trapezoidal rule on various $|z|$ values (less than 75) and with $\Delta\theta = \pi/1000$ gave accuracy in the seventh and eighth digit.

When using the recurrence formula for $I_n(y)$

$$G_{n-1}(y) - G_{n+1}(y) = \frac{2n}{y} G_n(y) ,$$

the summation

$$e^y = I_0(y) + 2 \sum_{n=1}^{\infty} I_n(y)$$

can be used to determine α. This formula is of course used in the same manner as are the $F_n(z)$ functions. That is,

$$G_{m+1}(y) = 0 ,$$

$$G_m(y) = \alpha ,$$

and, at the same n,

$$G_n(y) = \alpha I_n(y)$$

to any desired degree of accuracy, and

$$\alpha e^y = G_0(y) + \sum_{n=1}^{\infty} G_n(y).$$

No tests were made with this recurrence formula.

Determination of n versus m Values

We cannot give a definite value to the number of terms necessary in the recurrence to obtain good values for $J_n(z)$. It depends, of course, on z as well as the value given to a in $F_m(z) = a$, and also upon the particular accuracy desired.

The following empirical formulas will give values for m which generate a sequence $J_0(z)$, $\cdots$, $J_n(z)$, where $J_n(z)$ is of the magnitude 10^{-21}, that have the accuracies described in a later section. This is assuming computation is done in single precision and $a = 10^{-20}$.

$$m = 5.0\,|z| + 15 \quad \text{for} \quad 0.1 \le |z| < 10\ ; \tag{1}$$

$$m = 1.48\,|z| + 48 \quad \text{for} \quad 10 \le |z| < 150\ ; \tag{2}$$

$$m = 1.05\,|z| + 112 \quad \text{for} \quad 150 \le |z| \le 6000\ . \tag{3}$$

It is felt that Equation 3 can be extended indefinitely for all values of $z \ge 150$; however, tests did not go beyond the range given.

For all values of z and n the following general rule can be employed: *For single precision accuracy, a safe* m *vs* n *is obtained when the following condition is met:*

$$\frac{J_m(z)}{J_n(z)} < 10^{-5}\ .$$

Table 1 gives various n vs m values for real arguments in the range from 0.01 to 6000.

Table 1

The integral order n of the regular Bessel function having a particular magnitude in the range where $n > x$ is given for various x values, and the recurrence term m which will yield eight place accuracy for all orders from zero to n. Also tabulated is the value of m which yields overflow in the computer for the downward recurrence at $F_0(x)$. These values were obtained using double precision with $F_m(x) = 10^{-20}$.

Argument x	Magnitude of $J_n(x)$, where $n > x$, for particular orders n and recurrence terms m										m value yielding overflow
	10^{-2}		10^{-8}		10^{-16}		10^{-24}		10^{-32}		
	n	m	n	m	n	m	n	m	n	m	
.01	1	4	4	6	6	8	9	11	11	13	19
.05	1	5	5	7	8	10	11	13	14	16	22
.10	2	5	5	7	9	11	12	14	16	18	26
.50	3	7	8	11	13	15	17	19	21	23	34
2.0	6	12	12	16	19	22	25	28	30	33	47
5.0	9	13	18	23	26	30	34	38	40	43	60
8.0	14	16	22	28	32	37	41	45	48	51	70
12.0	17	20	28	34	39	44	48	53	57	61	81
20.0	25	29	39	46	51	57	62	67	72	76	100
40.0	46	55	63	72	79	86	92	98	104	109	138
100.0	109	119	130	143	152	163	170	179	186	193	229
300.0	310	321	334	361	374	389	398	411	420	431	485
500.0	511	530	555	583	587	604	615	631	642	655	715
1000.	1012	1047	1063	1088	1107	1128	1145	1164	1177	1192	1270
2000.	2015	2055	2080	2115	2135	2165	2185	2210	2225	2245	2340
3000.	3015	3065	3090	3127	3154	3208	3208	3235	3255	3277	3385
4000.	4015	4070	4095	4140	4170	4205	4230	4260	4280	4305	4425
6000.	6015	6080	6110	6160	6195	6235	6260	6295	6320	6350	6485

Details on Accuracy

z Real

It has been said that accuracy is gained in the downward recurrence until the point where oscillations begin to occur. If the correct n vs m has been used for a particular x, then at this n value $F_n(x) = \alpha J_n(x)$ to any desired degree of accuracy. From here down to $J_0(x)$, accuracy loss is due only to round off and truncation in the computer. But, in this region, the functions oscillate usually between ±0.3 and the number of accurate digits becomes more or less as the recurrence goes from larger to smaller numbers. Therefore, accuracy loss due to round off is not quite as bad as it might seem. In fact, when recurring for $x = 6000$, and $m = 6400$, the error only crept over to the fifth significant digit; this was using single precision in the computer.

Particular values were checked in already existing tables (References 3,4) for $0.05 \le x \le 100$ and $0 \le n \le 100$. Values having greater x and n were checked against values obtained using the recurrence in double precision. The accuracies obtained were the following:

Argument	Accuracy
$0.05 \le x \le 30$	within a few figures in the 8th digit
$30 < x \le 100$	at least 7 digits
$100 < x \le 1000$	at least 5 digits
$1000 < x \le 6000$	at least 4 digits

These accuracies are the worst obtained over the whole range of orders out to where $J_n(x) \approx 10^{-38}$.

Using double precision, of course, would give approximately eight more digits of accuracy for all arguments tabulated. It seemed unnecessary to check the recurrence for real arguments greater than 6000, in view of their sparse usage.

The accuracy stated above is obtained only if α in

$$F_n(x) = \alpha J_n(x)$$

is computed to at least the number of digits stated. Of course, if α is only good to a few digits then the Bessel function can be no better.

z Complex or Imaginary

In the downward recurrence for imaginary and complex argument $(x + iy)$, where $|y/x|$ is not small, the magnitudes of the numbers increase over the whole range of n values. Therefore, accuracy is gained or maintained over the whole range. These functions were checked in tables of Thomson functions for $\mathrm{ber}(x)$ and $\mathrm{bei}(x)$ and in formulas for $J_{1/2}(z)$, $J_{3/2}(z)$, $J_{5/2}(z)$. These all proved to be accurate to seven digits for various $|z|$ values ranging from .05 to 100.

For argument $(x + iy)$ where $|y/x|$ is small, (approximately, less than 0.1) the function behaves similarly to the real function in that the magnitudes oscillate and then continually decrease when n becomes greater than z. In general the accuracies should behave similarly to the accuracies of the real function.

Details on Use of the Recurrence

Choosing the Value for a

When using the above method for computing the regular Bessel function, the value we give to a is completely arbitrary. In other words the method "works" for any value of a. However, as a general rule, we should use as small a value, in magnitude, as is practical for the computer. This enables us to start out farther in the downward recurrence or use a larger m value and thereby obtaining higher order Bessel functions—and, of course, with less chance of overflow in recurring to numbers which are too large for the particular computer.

Consider a sequence of functions, say for real z,

$$J_0(x),\ J_1(x)\cdots J_n(x),\ J_{n+1}(x)\ .$$

The magnitudes of the functions may vary from approximately 0.1 for $J_0(x)$ down to say 10^{-38} for $J_n(x)$. That is, the values cover a range equal to approximately 10^{38}. Since the same range would be covered in our downward recurrence, the magnitude of $F_0(x)$ depends completely on the value of a at any particular m. If $a = 10^{-12}$, then $F_0(x)$ would be 38 orders of magnitude away or 10^{25}, and so on. Therefore, to compute larger order Bessel functions, a smaller value for

$$F_m(x) = a ,$$

should be used. It should be noted that the value given to a will not increase or decrease accuracy.

z Real

As has been stated, $J_0(x)$ does not become greater than 1 for all x, and

$$\lim_{n\to\infty} J_n(x) = 0 ,$$

the values oscillate usually between ±0.3 until n becomes greater than x, then they stay positive and begin to decrease at exponential rate to zero at $n = \infty$. The recurrence must start beyond this point for the method to succeed.

As a general rule, for all x, recurrence can begin at an m value for a particular order n where

$$\frac{F_m(x)}{F_n(x)} = \frac{J_m(x)}{J_n(x)} < 10^{-5} ,$$

and still obtain eight figure accuracy for $J_n(x)$.

z Complex

In contrast to the function of real argument,

$$|J_0(z)| \approx 0.1\, e^y ;$$

but still

$$\lim_{n\to\infty} J_n(x) = 0 ,$$

and the magnitude of the complex function decreases continually, slowly at first for low orders until, approximately when n becomes greater than x/2, they decrease at an exponential rate. The role that x plays will be discussed in a later section. Therefore the downward recurrence can begin anywhere in the range of n values as long as the condition

$$\frac{F_m(z)}{F_n(z)} < 10^{-5} ,$$

is met to give eight figure accuracy in $J_n(z)$.

z Imaginary

The function of imaginary argument behaves similarly to the function of complex argument in that they continually decrease in magnitude and, approximately at z/2, they decrease exponentially.

For integral orders, the functions oscillate between being pure real and pure imaginary; but for orders that are not integral, $J_n(iy)$ possess both real and imaginary parts. The integral orders are usually multiplied by $e^{-in\pi/2}$ to give the real function, $I_n(y)$, which is widely used. That is,

$$I_n(y) = e^{-in\pi/2} J_n(iy) .$$

More will be discussed on the function $I_n(y)$ in the next section. An example of the method will be found in Table 2, which gives $J_n(10)$ by the forward recurrence technique in both single and double precision, and by the backward recurrence technique in double precision.

Table 2
Values obtained for $J_n(10)$ from the forward recurrence in double and single precision and the backward recurrence in double precision.

n	$J_n(10)$ Forward Recurrence Single Precision	$J_n(10)$ Forward Recurrence Double Precision	$J_n(10)$ Backward Recurrence Double Precision
0	-0.24593576E-00	-0.24593576E-00	-0.24593576E-00
1	0.43472745E-01	0.43472745E-01	0.43472745E-01
2	0.25463031E-00	0.25463031E-00	0.25463031E-00
3	0.58379377E-01	0.58379379E-01	0.58379379E-01
4	-0.21960268E-00	-0.21960268E-00	-0.21960268E-00
5	-0.23406152E-00	-0.23406152E-00	-0.23406152E-00
6	-0.14458837E-01	-0.14458842E-01	-0.14458842E-01
7	0.21671092E-00	0.21671092E-00	0.21671092E-00
8	0.31785411E-00	0.31785412E-00	0.31785412E-00
9	0.29185566E-00	0.29185568E-00	0.29185568E-00
10	0.20748607E-00	0.20748610E-00	0.20748610E-00
11	0.12311649E-00	0.12311652E-00	0.12311652E-00
12	0.63370191E-01	0.63370255E-01	0.63370255E-01
13	0.28971968E-01	0.28972083E-01	0.28972083E-01
14	0.11956926E-01	0.11957163E-01	0.11957163E-01
15	0.45074252E-02	0.45079730E-02	0.45079730E-02
16	0.15653493E-02	0.15667561E-02	0.15667561E-02
17	0.50169241E-03	0.50564667E-03	0.50564667E-03
18	0.14040489E-03	0.15244248E-03	0.15244248E-03
19	0.37652134E-05	0.43146276E-04	0.43146277E-04
20	-0.12609708E-03	0.11513367E-04	0.11513369E-04
21	-0.50815354E-03	0.29071915E-05	0.29071994E-05
22	-0.20081478E-02	0.69683732E-06	0.69686851E-06
23	-0.83276965E-02	0.15889271E-06	0.15902198E-06
24	-0.36299255E-01	0.34069179E-07	0.34632629E-07
25	-0.16590872E-00	0.46393399E-08	0.72146349E-08
26	-0.79324437E-00	-0.10872479E-07	0.14405452E-08
27	-0.39589619E 01	-0.61176230E-07	0.27620052E-09
28	-0.20585150E 02	-0.31947916E-06	0.50937552E-10
29	-0.11131787E 03	-0.17279071E-05	0.90497669E-11
30	-0.62505851E 03	-0.97023820E-05	0.15510961E-11
31	-0.36390331E 04	-0.56486385E-04	0.25680948E-12
32	-0.21936946E 05	-0.34051320E-03	0.41122714E-13
33	-0.13675742E 06	-0.21227981E-02	0.63758926E-14
34	-0.88066203E 06	-0.13669954E-01	0.95817661E-15
35	-0.58517443E 07	-0.90832891E-01	0.13970838E-15
36	-0.40081548E 08	-0.62216029E 00	0.19782068E-16
37	-0.28273539E 09	-0.43887211E 01	0.27225057E-17
38	-0.20521604E 10	-0.31854376E 02	0.36447453E-18
39	-0.15313683E 11	-0.23770454E 03	0.47500695E-19
40	-0.11739457E 12	-0.18222410E 04	0.60308953E-20

NUMERICAL BEHAVIOR OF THE REGULAR BESSEL FUNCTION

In seeking to present a complete picture of the regular solutions to Bessel's equation

$$y'' + \frac{1}{z} y' + \left(1 - \frac{n^2}{z^2}\right) y = 0 ,$$

it seemed that a better understanding of the solutions was brought about when first they were observed as functions of n, and then as functions z. Then they could be brought together and a better picture could be visualized for all n and z.

z Real

Two-dimensional graphs have been drawn (Appendix B) showing various lines of constant magnitude for all real z and n for $0 < x \leq 6000$. These can be extended to larger x, since the lines are fairly linear.

It can be seen from a plot of $J_n(x)$ vs. n (Figure 1) that the functions oscillate between ±1, with the oscillations dampening in the range $0 \leq n \leq x$. Then for $n \geq x$ the values stay positive and begin to decrease very rapidly (here is where forward recurrence begins to fail), until at $n = \infty$, $J_n(x) = 0$. This is true for all values of x with the exception $J_n(0) = 0$, and $J_0(0) = 1$. Let us now consider the function more closely.

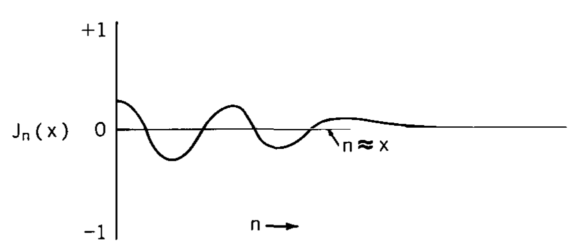


Figure 1—Plot of $J_n(x)$ vs. n, n continuous.

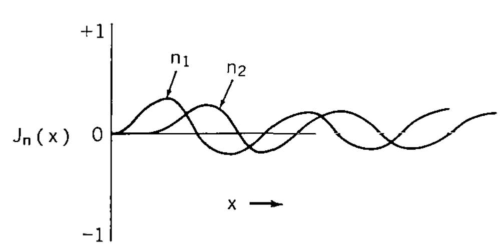
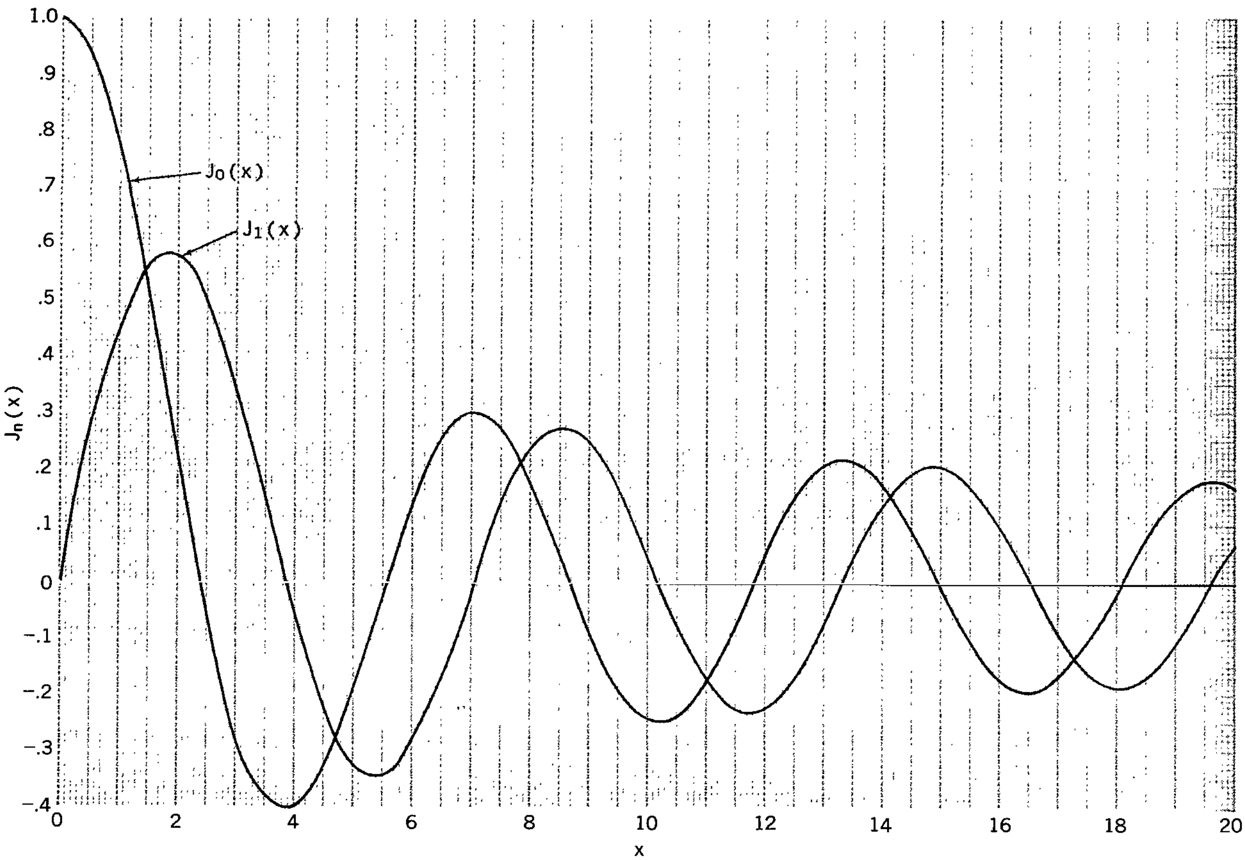


Figure 2—Plot of $J_n(x)$ vs. x, $n_1 < n_2$.

For $5 < x < n$, $J_n(x)$ oscillates between ±0.3 and for $0 < x \leq 5$, $n \leq 3$, $J_n(x)$ oscillates between ±0.2 and for all n, $J_n(n) \leq .1$. It can be seen that dampening of the oscillations takes longer for larger x, since it takes more terms for n to become similar to x.

On the other hand, it can be seen from a plot of $J_n(x)$ vs. x (Figure 2) that the functions start at $x = 0$, (except that $J_0(0) = 1$) and increase very slowly; the larger the n value the slower it increases, staying positive and never exceeding 0.1 until $n \approx x$. At this point they continue to increase, but as x increases oscillations begin. For $n > 3$ and all x, the oscillations are between ±0.3.

Graph I shows a detailed plot of $J_0(x)$ and $J_1(x)$ versus x.



Graph I—Detailed plot of $J_0(x)$ and $J_1(x)$ vs. x, $0 \leq x \leq 20$.

z Complex

The magnitude of the function for complex argument $z = x + iy$ begins at

$$|J_0(x + iy)| \approx 0.1\, e^y$$

for any z and, where $|y/x|$ is not small (see page 9)

$$\lim_{n \to \infty} J_n(x + iy) = 0 .$$

The functions continually decrease in magnitude as n becomes larger and, at approximately where $n > |z|/2$, they decrease at an exponential rate.

If we plot the modulus of $J_n(z)$ vs. n, we get the exponential curve shown in Figure 3. The effect of x in the argument is that for larger x, the gradient of the slope becomes less.

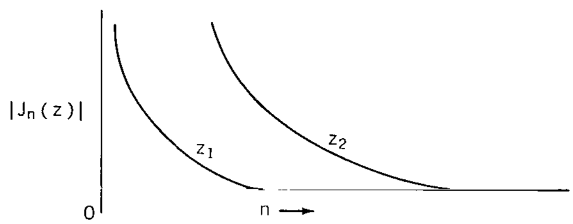


Figure 3—Plot of $|J_n(x + iy)|$ vs. n.

In considering a picture of the function of complex argument, it seems that the easiest and perhaps the best approach is set

$$J_n(x + iy) = u + iv$$

and plot the functions in both planes.

In Figure 4, the spiral is started at $J_0(x + iy)$, and as n increases the function spirals into the origin as a limit. Figure 5 is again a plot of $J_n(x + iy)$ as a function of n but now it is the three dimensional picture in x, iy, and n.

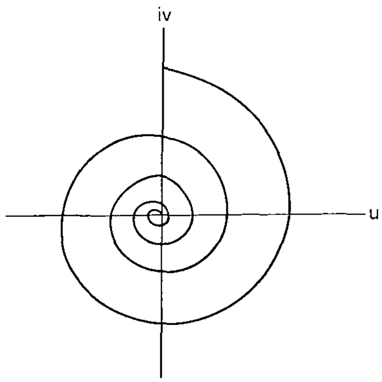


Figure 4—Projection of $J_n(x + iy)$ on the u, iv plane as a function of n, n continuous.

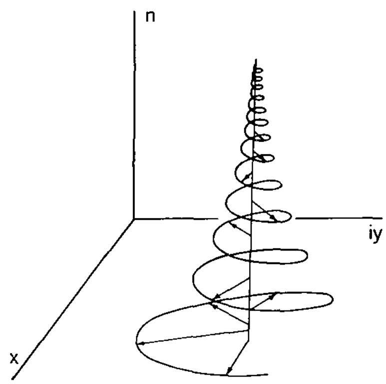


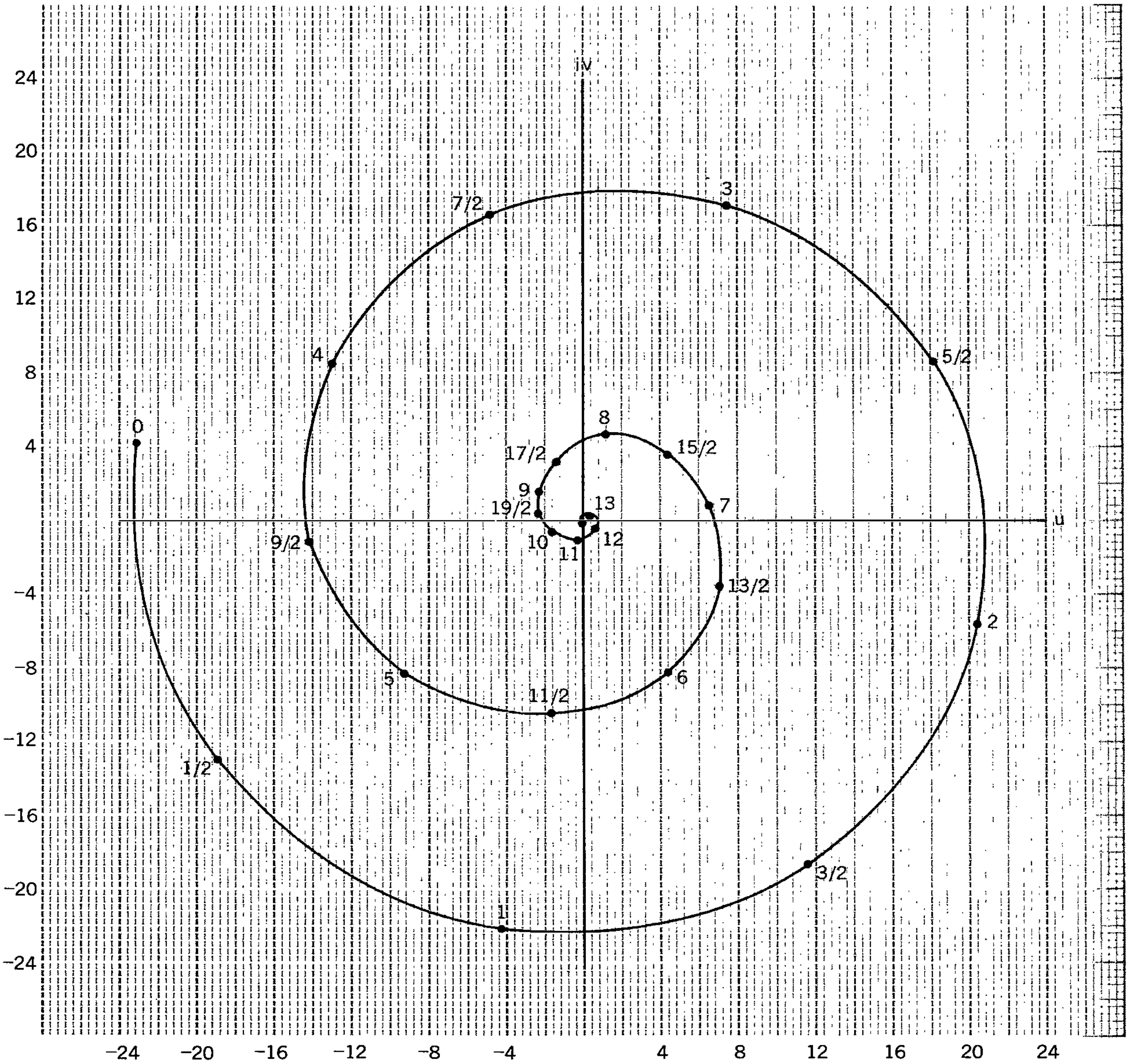
Figure 5—Plot of $J_n(x + iy)$ in the x, iy, n space as a function of n, n continuous.

If the spiral is moved around in the x, iy plane, the base of the spiral becomes larger as y increases, and becomes more spread out or flatter as x increases. If movement was directed along one of the 45 degree lines, the slope and the width would change at approximately the same rate. It should be noted that for movement along the 45 degree lines the plot would be of the Thomson Functions, $\mathrm{ber}_n(z)$, $\mathrm{bei}_n(z)$.

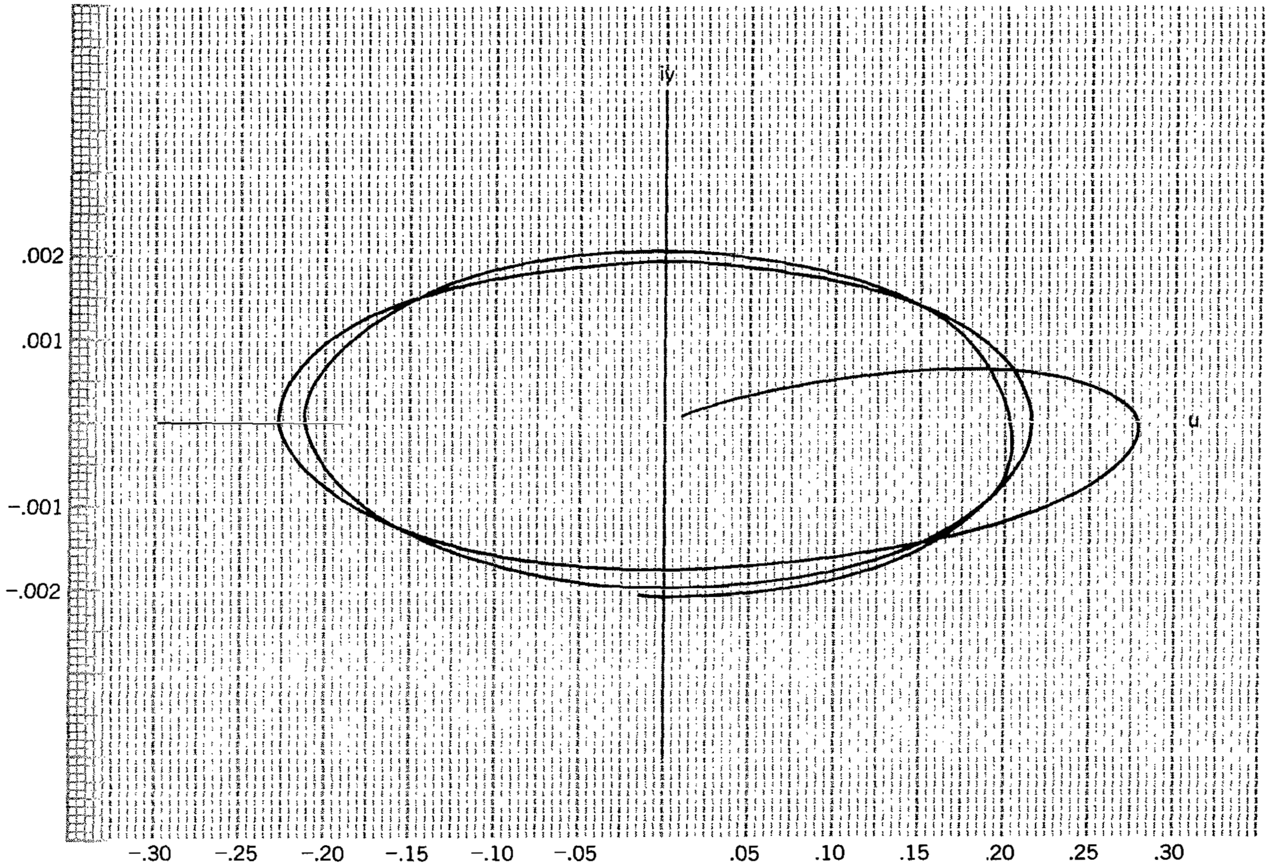
The direction of the spiraling is not significant, for it changes direction as the argument of z changes. The functions of arguments $(x + iy)$ and $(x - iy)$ would spiral into the origin in opposite directions.

If a three dimensional plot is made of $J_n(x + iy)$ in the x, iy plane, contour lines of constant magnitude would give the effect of a crater (Reference 5, page 127).

For argument $(x + iy)$ where $|y/x|$ is small, the modulus of $J_n(x + iy)$ oscillates until n becomes greater than z and at this point, it stays in the first quadrant of the u, iv plane, continually decreasing to the origin at $n = \infty$.



Graph II—Detailed plot of the projection of $J_n(10 + i\ 10)$ on the u, iv plane as a function of n.



Graph III—Detailed plot of the projection of J_n (15 + i.01) on the u, iv plane vs. n.

z Imaginary

The regular Bessel function of imaginary argument $J_n(iy)$ behaves similarly to the function of complex argument. $|J_0(iy)|$ is of the order of magnitude $0.1e^y$ with

$$\lim_{n\to\infty} J_n\,(iy) = 0\ ,$$

and the function spirals in toward the origin of the u, iv plane. The figures for imaginary argument would be similar to the complex argument with the integral orders lying on the axis and half orders displaced by an angle $\pi/4$, etc. In the u, iv plane they would appear as in Figure 6. Here again the direction of the spiral is determined by the argument of z, or $\pm\pi/2$. These functions are really solutions of the equation

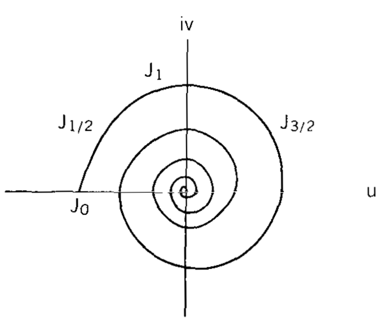


Figure 6—Projection of J_n (iy) on the u, iv plane.

$$y'' + \frac{1}{z}\,y' - \left(1 + \frac{n^2}{z^2}\right)y = 0\ ,$$

which occurs frequently in problems of mathematical physics; however, it is usually desirable to express the solutions in real form.

The solutions $J_n(iy)$, $Y_n(iy)$ are not always real, as was shown for the regular function; however the function $e^{-in\pi/2}J_n(iy)$ is always real and is a solution of the equation. In effect the imaginary values are rotated thru an angle of $\pi/2$. These functions are the $I_n(y)$ functions and are defined by

$$I_n(y) = e^{-in\pi/2} J_n(iy) ;$$

they obey the recurrence formula

$$I_{n-1}(y) - I_{n+1}(y) = \frac{2n}{y} I_n(y)$$

the use of which was discussed in a previous section.

CONCLUDING REMARKS

It has been shown that the backward recurrence is an accurate and especially easy method for computing the regular Bessel Function. Indirectly, this report came about through a need for and investigation of computing the 1/2 order Bessel functions for the intensity functions of light scattering by spherical particles. Since the calculations required the Bessel functions varying in the order, the recurrence was especially suitable. The backward recurrence method should be a very useful tool to those requiring Bessel functions of any order or argument:

The magnitude graphs (Appendix B) are significant in that they give an overall picture of the functions varying in both the order and the argument. This could be useful in many computations, e.g., an integral of the type

$$A = \int_a^b J_n(x)\,dx .$$

Insight might be obtained as to the range of x that would add to the value of the integral and so on.

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Appendix A

Solutions of Bessel's Differential Equation

If α is any real constant, Bessel's equation of order α is

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\alpha^2}{x^2}\right)y = 0 .$$

This is a linear differential equation of second order, and therefore has two and only two independent solutions. If these two solutions are $y_1(x)$ and $y_2(x)$, then there cannot exist two nonzero constants c_1 and c_2 such that

$$c_1 y_1(x) + c_2 y_2(x) = 0 .$$

That is to say that any other solution, say $y_3(x)$, is a linear combination of the two independent solutions, or

$$y_3(x) = c_1 y_1(x) + c_2 y_2(x) .$$

Note that Bessel's equation has a singular point at $x = 0$, but a solution can be obtained by the method of Frobenius. That is, assume a solution of the type

$$y = \sum_{k=0}^{\infty} C_k x^{k+\alpha}$$

and it turns out that the two solution types are completely dependent upon α. They are referred to by order α and argument x. If α is an integer, the two independent solutions are $J_\alpha(x)$ and $Y_\alpha(x)$, α = integer. If α is not an integer the two solutions are $J_\alpha(x)$ and $J_{-\alpha}(x)$, $\alpha \neq$ integer. If α is half an odd integer they are usually referred to as the *plus and minus half orders*. $J_\alpha(x)$ is the regular solution; and $J_{-\alpha}(x)$, α not an integer, and $Y_\alpha(x)$, α = integer, are the irregular solutions. The two solutions are also referred to as Bessel functions of the first and second kind respectively.

If the argument x is purely imaginary, the solution is referred to as

$$I_\alpha(x) \quad \text{and} \quad K_\alpha(x) .$$

$I_\alpha(x)$ is analogous to the regular Bessel function and $K_\alpha(x)$ to the irregular Bessel function.

$H_\alpha{}^{(1)}(x)$ and $H_\alpha{}^{(2)}(x)$ are known as Bessel functions of the third kind, or first and second Hankel Functions respectively. They are defined as

$$H_\alpha^{(1)}(x) = J_\alpha(x) + i\, Y_\alpha(x) \,, \tag{A1}$$

$$H_\alpha^{(2)}(x) = J_\alpha(x) - i\, Y_\alpha(x) \,. \tag{A2}$$

These are obtained from

$$F_\alpha(x) = AJ_\alpha(x) + BY_\alpha(x)$$

with $A = 1$ and $B = \pm i$. If we add and subtract Equations A1 and A2, we arrive at other identities which are sometimes useful:

$$J_\alpha(x) = \frac{1}{2}\left(H_\alpha^{(1)}(x) + H_\alpha^{(2)}(x)\right) \,,$$

$$Y_\alpha(x) = \frac{-i}{2}\left(H_\alpha^{(1)}(x) - H_\alpha^{(2)}(x)\right) \,.$$

For complex argument $i^{3/2}x$, one obtains the Thomson function $\mathrm{ber}_\alpha(x)$, $\mathrm{bei}_\alpha(x)$:

$$J_\alpha\left(i^{3/2}x\right) = \mathrm{ber}_\alpha(x) + i\,\mathrm{bei}_\alpha(x) \,.$$

If $\alpha = 0$ the subscripts are usually omitted. Listed below are the two general types of differential equations and their associated functions as solutions:

$$y'' + \frac{1}{x}y' + k^2 y = 0 \qquad \left\{\begin{aligned} y &= AJ_0(kx) + BY_0(kx) \\ y &= AH_0^{(1)}(kx) + BH_0^{(2)}(kx) \end{aligned}\right\};$$

$$y'' + \frac{1}{x}y' + \left(k^2 - \frac{\alpha^2}{x^2}\right)y = 0 \qquad \left\{\begin{aligned} y &= AJ_\alpha(kx) + BY_\alpha(kx) \\ y &= AJ_\alpha(kx) + BJ_{-\alpha}(kx) \\ y &= AH_\alpha^{(1)}(kx) + BH_\alpha^{(2)}(kx) \end{aligned}\right\};$$

$$y'' + \frac{1}{x}y' - k^2 y = 0 \qquad \left\{ y = AI_0(kx) + BK_0(kx) \right\};$$

$$y'' + \frac{1}{x}y' - \left(k^2 - \frac{\alpha^2}{x^2}\right)y = 0 \qquad \left\{\begin{aligned} y &= AI_\alpha(kx) + BK_\alpha(kx) \\ y &= AI_\alpha(kx) + BI_{-\alpha}(kx) \end{aligned}\right\}.$$

Appendix B

The Magnitude Graphs

The Magnitude graphs are plots of constant $J_n(x)$ in the x, n plane. The plot of $J_n(x)$ is continuous in this plane both as a function of n and as a function of x. It is a rolling surface and slopes down into the surface of the graph as x and n become larger. Between the zeros, lines of constant magnitude can be drawn, of which there would be two of the same value that connect at some larger x and n value; however, it was felt these would be superfluous here and they were left out. Since the surface is continuous, peak values either plus or minus would fall between two zeros and the magnitude of the function would decrease as we go toward the zero. All lines below the 10^{-2} line represent the approximate n and x value for the zeros of the Bessel function with the sign of the function indicated between the zeros.

In looking at the magnitude graphs that follow, pick a particular case, say $J_n(x)$ varying in n for x = 40; it can be seen that $J_0(40)$ is positive and there are six oscillations before the functions stay positive. It can be seen that

$$-0.3 \le J_n(x) \le 0.3 \quad , \qquad 0 \le n \le 46 \quad ,$$

$$0 < J_n(x) < 10^{-2} \quad , \qquad n > 46 \quad ,$$

$$0 \le J_n(x) \le 10^{-16} \quad , \qquad n > 79 \quad ,$$

and so on. A graph of $J_n(40)$ versus n would appear as in Figure B1.

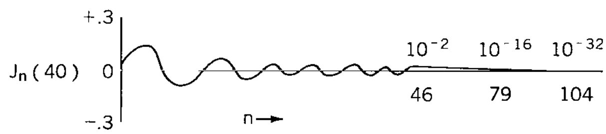


Figure B1—Plot of $J_n(40)$ vs. n.

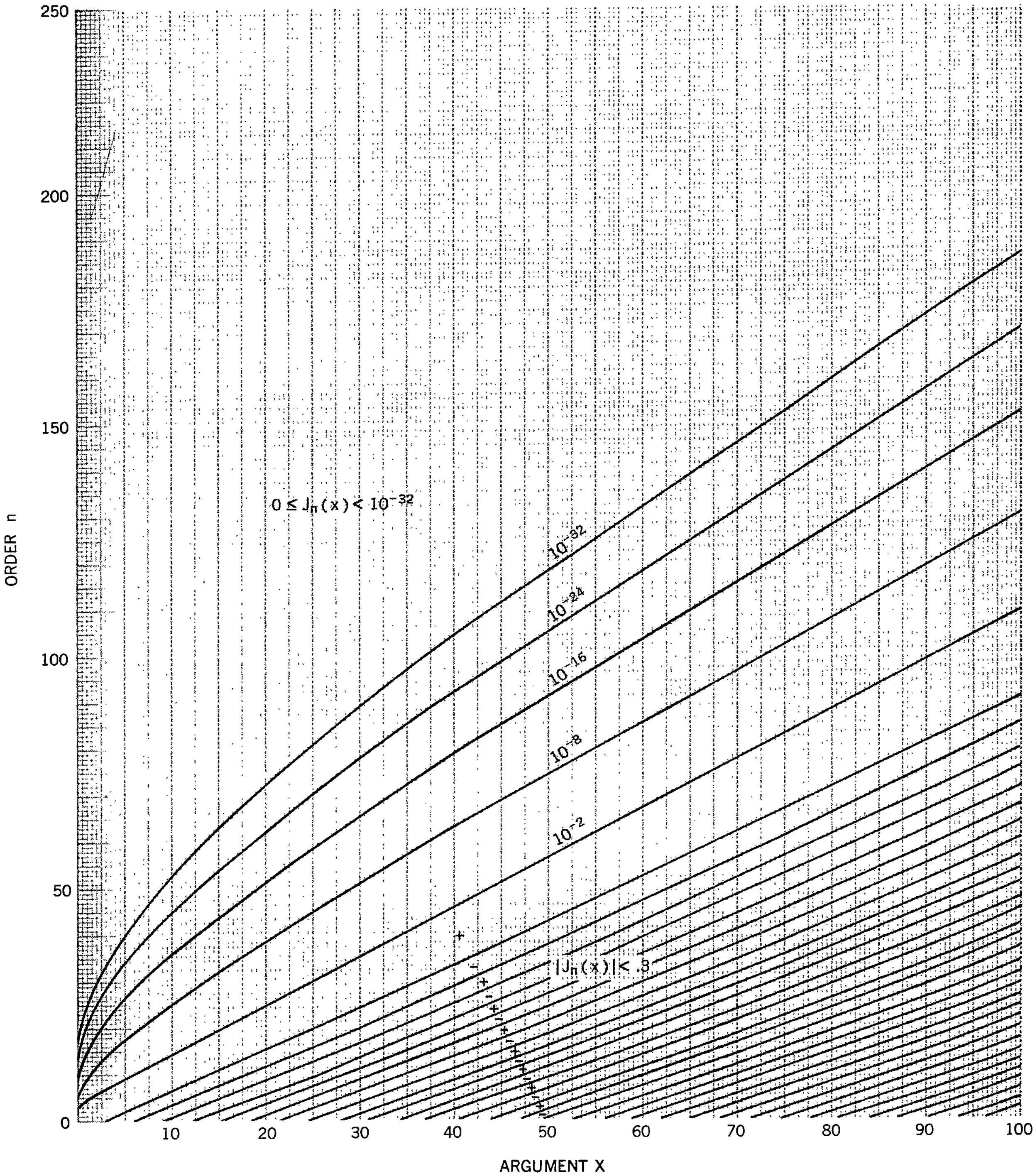
On the other hand, looking at $J_n(x)$ varying in x, say for n = 50, the function increases from 0 to 10^{-32} for x ranging from 0 to approximately 8.8, and oscillations do not begin until x is approximately 55. It can be seen that

$$J_{50}(x) < 10^{-32} \quad , \qquad 0 \le x < 8.8 \quad ,$$

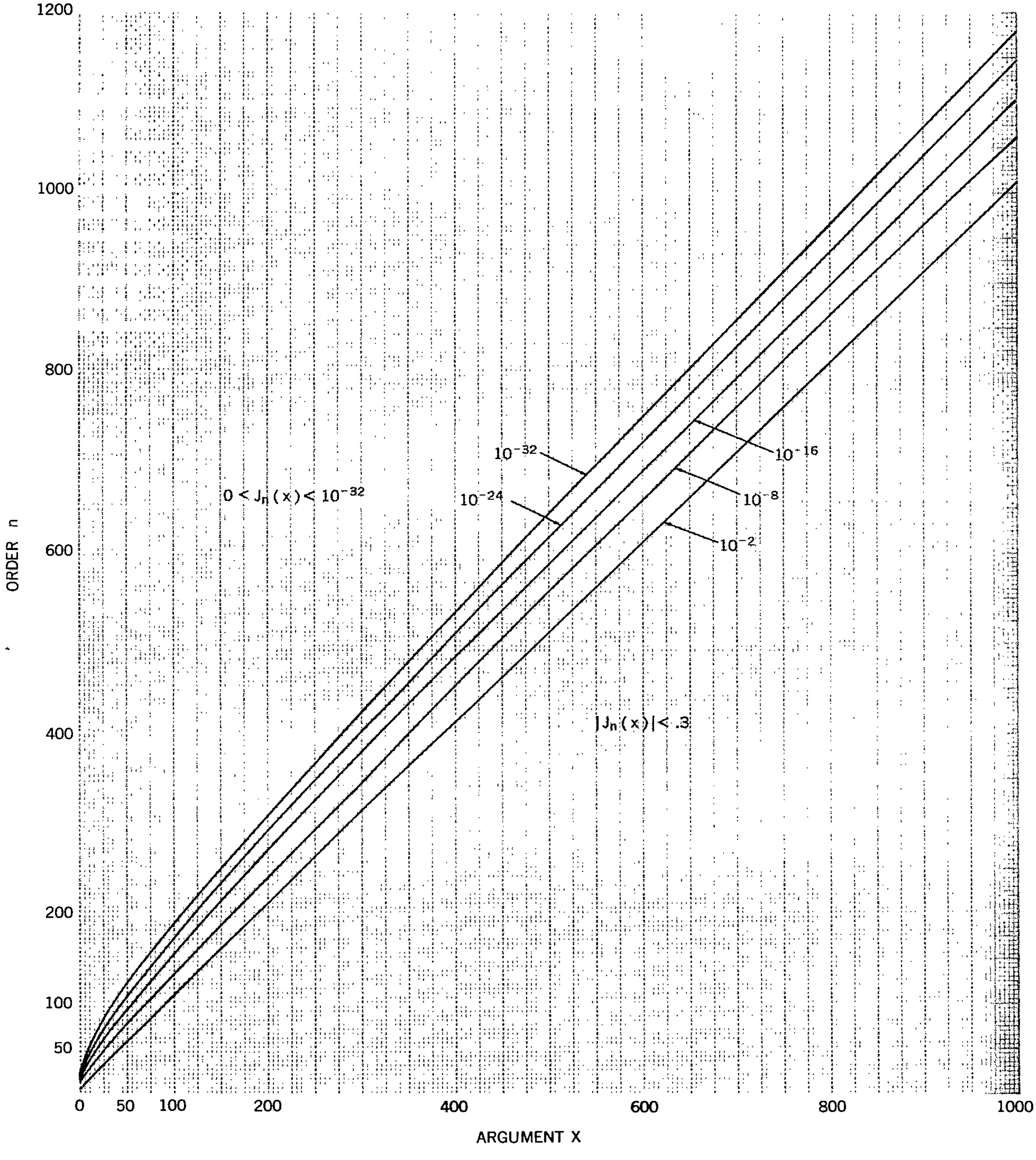
$$10^{-32} \le J_{50}(x) < 10^{-2} \quad , \qquad 8.8 \le x \le 44 \quad ,$$

$$-0.3 \le J_{50}(x) \le 0.3 \quad , \qquad x > 44 \quad .$$

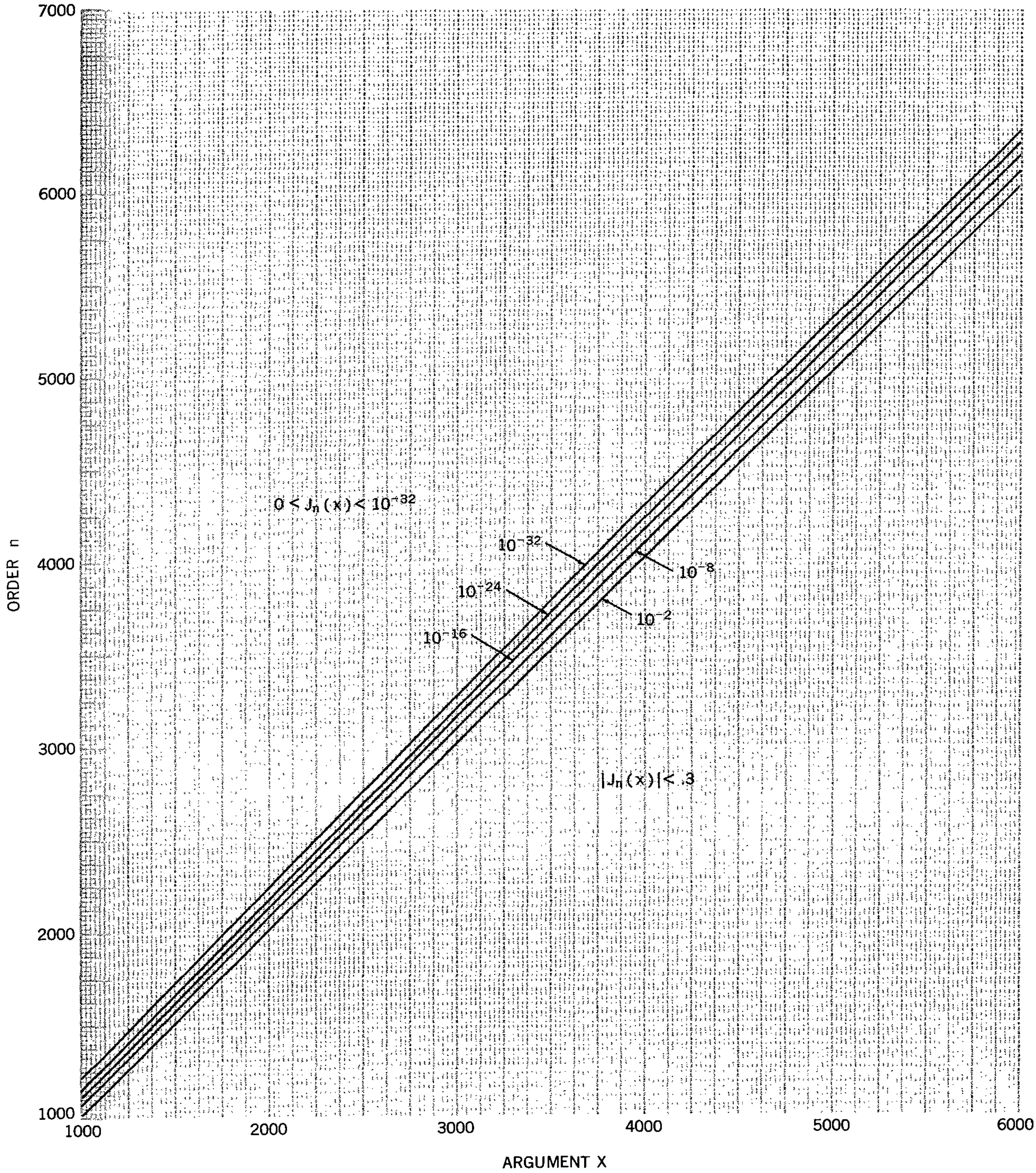
Therefore, an idea of how the function behaves can be obtained for any order or argument within the limits of the graph.



250
200
150
100
50
0
ORDER n
$0 \leq J_n(x) < 10^{-32}$
10^{-32}
10^{-24}
10^{-16}
10^{-8}
10^{-2}
$|J_n(x)| < .3$
10
20
30
40
50
60
70
80
90
100
ARGUMENT X



1200
1000
800
600
400
200
100
50
ORDER n
0
50
100
200
400
600
800
1000
ARGUMENT X
0 < Jn(x) < 10^-32
|Jn(x)| < .3
10^-32
10^-24
10^-16
10^-8
10^-2



NASA-Langley, 1964 G-494